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Compressible flow equations based on moving coordinates determined by the wave speed

Zheng Chen[‡], Bo Gao and Zi-Niu Wu[*,†]

Department of Engineering Mechanics, Tsinghua University, Beijing 100084, People's Republic of China

SUMMARY

In this paper, we consider moving coordinates with a speed determined by the wave speed. This allows us to move the coordinates at the speed of contact discontinuities, expansion fans, or other wave structures. While recognizing that the Lagrangian coordinates, which can well resolve the sliplines, behave badly for expansion waves, we demonstrate that a coordinate system moving at the characteristic speed of the expansion fan behaves much better for expansion waves. Moreover, the new coordinate system allows one to well capture the shock wave, the slipline and the expansion waves at the same time. Copyright © 2006 John Wiley & Sons, Ltd.

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1. INTRODUCTION

There are two classical coordinate systems used to describe fluid flows: the Eulerian coordinate system and the Lagrangian coordinate system. In the Eulerian approach, one considers what happens at every fixed point in space as a function of time. The velocities and the other properties of fluid elements are considered to be functions of time and fixed space coordinates. In the Lagrangian approach, one looks for the dynamic history of each selected fluid element. The positions of fluid particles and the other properties are considered to be functions of the time and their initial positions. Both approaches have some advantages in classical fluid mechanics. The different

*Correspondence to: Zi-Niu Wu, Department of Engineering Mechanics, Tsinghua University, Beijing 100084, People's Republic of China.
†E-mail: ziniuwu@tsinghua.edu.cn
‡Actual address: Princeton University, U.S.A.

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coordinate systems are regarded as equivalent to each other for classical fluid mechanics [1]. But the Lagrangian approach gives more information about the history of the fluid particles and is of great theoretical interest. Nowadays, the Lagrangian approach still attracts great attentions from mathematicians. For instance, Despand and Mazeran [2] proposed a new and canonical way of writing the equations of gas dynamics in Lagrangian coordinates in two dimensions as a weakly hyperbolic system of conservation laws.

Hui and his co-workers [3, 4], originally proposed a unified coordinate system which includes the Eulerian approach and Lagrangian approach as two special cases. It keeps the advantages of the Lagrangian approach in capturing sliplines while avoiding severe grid deformation. Besides these advantages, Gao and Wu [5] found a very useful application of this method to automatically generate grid during the computation. With the unified coordinate system, one simply builds a narrow grid near the inflow boundary. Then by continuously injecting new columns of grid, the grid will gradually fill up the entire domain. In this way, the grid is automatically and physically generated during the computation.

Contact discontinuities, expansion fans, and shock waves are fundamental waves in compressible flows. Contact discontinuities are linearly degenerate waves which move at the convective velocity of fluid particles so that it can be well captured by using the Lagrangian coordinate system when the equations are solved numerically by using computational fluid dynamics. Expansion fans move at a speed which is a linear combination of the convective speed u and sound speed a. To some extent the speed of shock waves can also be considered as a combination of convective speed and sound speed with variable coefficients. For convenience we define

$$B = \alpha u + \beta a \tag{1}$$

as the *characteristic wave speed*, where α and β are two parameters. If we take $\alpha = 1$ and $\beta = 0$, then B is the speed of contact discontinuities. If $\alpha = 1$ and $\beta = \pm 1$, then B is the speed of expansion fans.

Inspired by the advantage of Lagrangian coordinate system for capturing contact discontinuities, in this paper we consider compressible flow equations written in a *wave frame*, defined as the frame moving at the characteristic wave speed B. The wave frame can be alternatively called generalized characteristic coordinate system (GCCS) since B as defined by (1) is a generalized definition of characteristics. We recover the classical Eulerian approach for $\alpha = 0$ and $\beta = 0$, the classical Lagrangian approach for $\alpha = 1$ and $\beta = 0$, and the unified coordinate system of Hui *et al.* [3] for $\beta = 0$ and $0 \leqslant \alpha = h \leqslant 1$.

We are interested in the fundamental properties of the compressible flow equations in the wave frame. Notably, we will study the solution of the Riemann problem in the wave frame and the equivalence of Riemann problems on both the wave frame and the original Eulerian coordinate system. The equivalence between an expansion fan in different classical coordinate systems has been studied in Reference [6] where it was shown that an expansion fan can degenerate to a linear wave, become a compressible wave, or remain to be an expansion wave, when the original system is replaced by another coordinate system. In Reference [7], the equivalence of weak solutions between two classical coordinate systems is also studied.

This paper will be organized as follows. In Section 2, we introduce the wave frame and rewrite the compressible flow equations in the wave frame. The characteristics and hyperbolicity of the system are analysed. In Sections 3 and 4 we discuss simple waves, shock waves and Riemann problem in the wave frame. The equivalence of the expansion fans and shock waves in the wave

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frame and in the physical space is studied in Sections 5 and 6. Conclusions and preliminary numerical results will be given in Section 7.

2. THE COMPRESSIBLE FLOW EQUATIONS IN THE WAVE FRAME

2.1. The compressible flow equations in the classical Eulerian coordinate system

In this subsection, we recall some known properties of compressible flow equations in the Eulerian system.

Consider the one-dimensional Euler equations in gas dynamics, which, in the original (Eulerian) coordinate system (x, t), can be written in the following conservative form:

$$w_t + f(w)_x = 0 \tag{2}$$

with

$$w = (\rho, \rho u, \rho E)^t$$
$$f(w) = \left(\rho u, \rho u^2 + p, \rho u\left(E + \frac{p}{\rho}\right)\right)^t$$

Here ρ is the density, u is the velocity of the fluid particle, E is the total energy, $p = (\gamma - 1)(\rho E - \frac{1}{2}\rho u^2)$ is the pressure, and γ is the ratio between the specific heats at constant pressure and constant volume. The sound speed is defined by $a = \sqrt{\gamma p/\rho}$.

System (2) is hyperbolic, i.e. the eigenvalues of the Jacobian matrix $C = \mathrm{d}f(w)/\mathrm{d}w$ are all real and there exist a complete set of eigenvectors for C. The three eigenvalues of C are

$$\mu_1 = u, \quad \mu_2 = \mu^+ = u + a, \quad \mu_3 = \mu^- = u - a$$

The wave corresponding to the eigenvalue $\mu = \mu^+$ or μ^- is called an expansion wave in the physical space if $\partial\mu/\partial x > 0$, and a compression wave if $\partial\mu/\partial x < 0$, and a linearly degenerate one if $\partial\mu/\partial x = 0$.

On the characteristic plane (x, t), the characteristics $\mathrm{d}x/\mathrm{d}t = \mu$ diverge for an expansion wave and converge (up to the formation of a shock) for a compression wave. A linearly degenerate wave neither diverges nor converges on the characteristic plane.

Let (x_c, t_c) be the centre of an expansion fan. For a left-going expansion fan, the head and the tail of the expansion move at a velocity $u_\mathrm{l} - a_\mathrm{l}$ and $u_\mathrm{r} - a_\mathrm{r}$, respectively. Inside the expansion fan, both the entropy $S = p/\rho^\gamma$ and the Riemann invariant $R^+ = a + [(\gamma - 1)/2]u$ remain constant. The solution in the left-going expansion fan is given by

$$\frac{x - x_c}{t - t_c} = u - a \tag{3}$$

$$u(x, t) = \frac{2}{\gamma + 1}\left[R^+ + \frac{x - x_c}{t - t_c}\right] \tag{4}$$

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$$a(x,t) = \frac{2}{\gamma+1}\left[R^+ - \frac{\gamma-1}{2}\frac{x-x_c}{t-t_c}\right] \tag{5}$$

$$\rho = \rho_1\left(\frac{a}{a_1}\right)^{2/(\gamma-1)} = \rho_1\left(\frac{\frac{2}{\gamma+1}\left[R^+ - \frac{\gamma-1}{2}\frac{x-x_c}{t-t_c}\right]}{a_1}\right)^{2/(\gamma-1)} \tag{6}$$

For a right-going expansion fan, this solution becomes

$$\frac{x-x_c}{t-t_c} = u + a \tag{7}$$

$$u(x,t) = -\frac{2}{\gamma+1}\left[R^- - \frac{x-x_c}{t-t_c}\right] \tag{8}$$

$$a(x,t) = \frac{2}{\gamma+1}\left[R^- + \frac{\gamma-1}{2}\frac{x-x_c}{t-t_c}\right] \tag{9}$$

$$\rho = \rho_1\left(\frac{a}{a_1}\right)^{2/(\gamma-1)} = \rho_1\left(\frac{\frac{2}{\gamma+1}\left[R^- + \frac{\gamma-1}{2}\frac{x-x_c}{t-t_c}\right]}{a_1}\right)^{2/(\gamma-1)} \tag{10}$$

Now we consider shock waves. Let w_0 and w be the post-shock and post-shock state. We use $\langle w\rangle = w - w_0$ to denote the jump across the discontinuity. Then in the physical space the following Rankine–Hugoniot relation is satisfied for both shock wave and contact discontinuity

$$\langle f(w)\rangle = s'\langle w\rangle, \quad x = x_s \tag{11}$$

where x_s denotes the position of the discontinuity and $s' = \mathrm{d}x_s/\mathrm{d}t$ is the speed of the shock wave.

2.2. *Conservation form of the compressible flow equations in the wave frame*

The GCCS (ξ, λ) is related to the original system (x, t) by

$$\mathrm{d}t = \mathrm{d}\lambda \tag{12}$$

$$\mathrm{d}x = A\,\mathrm{d}\xi + B\,\mathrm{d}\lambda \tag{13}$$

where B is defined by

$$B = \alpha u + \beta a \tag{14}$$

and A must satisfy the Cauchy–Riemann relation for $\mathrm{d}x$ to be a full differential

$$\frac{\partial A}{\partial \lambda} = \frac{\partial B}{\partial \xi} \tag{15}$$

which is also called geometrical conservation law [3].

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Following Viviand [8], system (2) in the transformed frame can be written in the following conservative form:

$$V_\lambda + F(V)_\xi = 0 \tag{16}$$

with

$$V = Aw, \quad F(V) = -Bw + f(w)$$

The physical conservation law (16) and the geometrical conservation law (15) must be solved simultaneously. Following Hui *et al.* [3], we unify both conservation laws by writing

$$W_\lambda + F(W)_\xi = 0 \tag{17}$$

where

$$W = \begin{pmatrix} \rho A \\ \rho u A \\ \rho E A \\ A \end{pmatrix}, \quad F(W) = \begin{pmatrix} \rho u - \rho B \\ \rho u^2 + p - \rho u B \\ \rho u \left(E + \dfrac{p}{\rho} \right) - \rho E B \\ -B \end{pmatrix}$$

2.3. *Characteristics in the wave frame*

It turns out to be extremely tedious and cumbersome if we directly work with (17) to study the characteristics. To be more efficient, we introduce a new set of variables defined by $U = (\rho, u, p, A)$ so that (17) can be rewritten as

$$\frac{\partial U}{\partial \lambda} + C \frac{\partial U}{\partial \xi} = 0 \tag{18}$$

with

$$C = \begin{pmatrix} \dfrac{u-B}{A} & \dfrac{\rho}{A} & 0 & 0 \\ 0 & \dfrac{u-B}{A} & \dfrac{1}{\rho A} & 0 \\ 0 & \dfrac{\rho a^2}{A} & \dfrac{u-B}{A} & 0 \\ \dfrac{\beta a}{2\rho} & -\alpha & -\dfrac{\beta a}{2p} & 0 \end{pmatrix} \tag{19}$$

The eigenvalues of (19) are found to be

$$\mu_1 = \frac{u-B-a}{A}, \quad \mu_2 = \frac{u-B}{A}, \quad \mu_3 = \frac{u-B+a}{A}, \quad \mu_4 = 0$$

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and the corresponding right eigenvectors are (in case $B \neq u, u \pm a$)

$$R_1 = \begin{pmatrix} -\rho a^{-1} \\ 1 \\ -\rho a \\ -\frac{A}{2}\frac{2\alpha+\beta(1-\gamma)}{u-B-a} \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \frac{\beta a A}{2\rho(u-B)} \end{pmatrix}$$

$$R_3 = \begin{pmatrix} \rho a^{-1} \\ 1 \\ \rho a \\ -\frac{A}{2}\frac{2\alpha-\beta(1-\gamma)}{u-B+a} \end{pmatrix}, \quad R_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Let us denote $\nabla = (\partial/\partial\rho, \partial/\partial u, \partial/\partial p, \partial/\partial A)$ and study the four characteristic fields.

(1) For the first characteristic field, we have $\nabla\mu_1 \cdot R_1 = (1+\gamma)/2A \neq 0$. Hence, the first characteristic field is genuinely nonlinear.
(2) For the second, we have $\nabla\mu_2 \cdot R_2 = 0$. Hence, the second characteristic field is linearly degenerate.
(3) For the third, we have $\nabla\mu_3 \cdot R_3 = (1+\gamma)/2A \neq 0$. Hence, the third characteristic field is genuinely nonlinear.
(4) For the fourth, $\mu_4 = 0$, we have $\nabla\mu_4 \cdot R_4 = 0$. Hence, the fourth characteristic field is linearly degenerate.

2.4. Hyperbolicity in the wave frame

Obviously, two of the eigenvalues can be equal, under the condition that $u = B$ or $B + a$ or $B - a$. This condition is possible if α and β satisfy

$$(1-\alpha)M_{\text{loc}} = \beta \quad \text{or} \quad \beta + 1 \quad \text{or} \quad \beta - 1 \tag{20}$$

where $M_{\text{loc}} = u/a$ is the local Mach number.

Proposition 1
System (18), and hence (17), is hyperbolic if condition (20) is not satisfied. The system becomes weakly hyperbolic if condition (20) is satisfied.

Hui *et al.* [3] proved that the Euler equations using the Lagrangian coordinates are only weakly hyperbolic. In fact, for Lagrangian coordinates, $\alpha = 1$ and $\beta = 0$ so that (20) is satisfied. If we want to recover strong hyperbolicity, one can simply use a small β to ensure $(1-\alpha)M_{\text{loc}} \neq \beta$ for $\alpha \to 1$. This is more explicitly stated in the following proposition.

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Proposition 2
To recover strong hyperbolicity of the original Lagrangian approach with $B = u$, it is sufficient to perturb B to $B = u + \beta a$ for some small parameter $\beta \neq 0$.

The wave corresponding to the eigenvalue $\mu = \mu_1$, μ_2, μ_3, or μ_4 is called an expansion wave in the wave frame if $\partial\mu/\partial\xi > 0$, a compression wave if $\partial\mu/\partial\xi < 0$, and a linearly degenerate one if $\partial\mu/\partial\xi = 0$.

On the characteristic plane (ξ, λ), the characteristics $\mathrm{d}\xi/\mathrm{d}\lambda = \mu$ diverge for an expansion wave and converge (up to the formation of a shock) for a compression wave. A linearly degenerate wave neither diverges nor converges on the characteristic plane.

3. SIMPLE WAVES AND SHOCK WAVES IN THE WAVE FRAME

3.1. Simple waves

Simple waves in the physical space are expansion fans. Now we want to determine the simple waves in the wave frame. To recover strong hyperbolicity, we only consider cases in which condition (20) is not satisfied.

For the kth characteristic field, let $r_k^{(i)}$ be the ith component of the right eigenvector R_k. Also, let $u^{(i)}$ be the ith component of U. Then the kth generalized Riemann invariants can be obtained by integrating

$$\frac{\mathrm{d}u}{\mathrm{d}\rho} = \frac{r_k^{(2)}}{r_k^{(1)}}, \quad \frac{\mathrm{d}p}{\mathrm{d}\rho} = \frac{r_k^{(3)}}{r_k^{(1)}}, \quad \frac{\mathrm{d}A}{\mathrm{d}\rho} = \frac{r_k^{(4)}}{r_k^{(1)}}$$

For convenience, we use (ρ_0, u_0, p_0, A_0) to denote either $(\rho_\mathrm{l}, u_\mathrm{l}, p_\mathrm{l}, A_\mathrm{l})$ or $(\rho_\mathrm{r}, u_\mathrm{r}, p_\mathrm{r}, A_\mathrm{r})$.

(1) For the first characteristic field, which is genuinely nonlinear as we have shown, we have

$$u = u_0 - \frac{2a_0}{\gamma - 1}\left[\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - 1\right] \tag{21}$$

$$a = a_0\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} \tag{22}$$

$$p = p_0\left(\frac{\rho}{\rho_0}\right)^{\gamma} \tag{23}$$

$$A = A_0\left(\frac{(1-\alpha)\left(u_0 + \frac{2a_0}{\gamma-1}\right) - a_0\left(\frac{2(1-\alpha)}{\gamma-1} + 1 + \beta\right)\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2}}{u_0(1-\alpha) - a_0\,(1+\beta)}\right)^{-\frac{2\alpha+\beta(1-\gamma)}{2(1-\alpha)+(\gamma-1)(1+\beta)}} \tag{24}$$

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Consider a 1-simple wave centred at $(\lambda, \xi) = (0, 0)$. Since the second characteristic curve is a straight line in a 1-simple wave, we can solve $\mathrm{d}\xi/\mathrm{d}\lambda = (u - B - a)/A$ to get

$$\frac{(1-\alpha)u - (1+\beta)a}{A} = \frac{\xi}{\lambda} \tag{25}$$

Inserting (21)–(24) into (25) yields the following relation:

$$\frac{\rho}{\rho_0} = \left\{ \frac{1}{\dfrac{2a_0(1-\alpha)}{\gamma - 1} + a_0(1+\beta)} \left\{ (1-\alpha)\left(u_0 + \frac{2a_0}{\gamma-1}\right) - \left[A_0(u_0(1-\alpha) - a_0(1+\beta))^{[2\alpha+\beta(1-\gamma)]/[2(1-\alpha)+(\gamma-1)(1+\beta)]} \frac{\xi}{\lambda} \right]^{[2(1-\alpha)+(\gamma-1)(1+\beta)]/(1+\gamma)} \right\} \right\}^{2/(\gamma-1)} \tag{26}$$

Hence, with (26) and (21)–(24), we can obtain (ρ, u, p, A) at any point (ξ, λ) inside the first characteristic field.

Proposition 3

In the case of $A \equiv 1$, $B \equiv 0$, (26) reduces to

$$\frac{\rho}{\rho_0} = \left\{ \frac{\gamma - 1}{\gamma + 1} \frac{1}{a_0} \left[\left(u_0 + \frac{2a_0}{\gamma - 1} \right) - (u - a) \right] \right\}^{2/(\gamma-1)}$$

which reduces to the relation as in the Eulerian approach.

(2) For the second characteristic field, which is linearly degenerate as we have shown, we have

$$\begin{aligned} u &= u_0 \\ p &= p_0 \\ A &= A_0 \frac{(1-\alpha)u_0 - \beta\sqrt{\dfrac{\gamma p_0}{\rho}}}{(1-\alpha)u_0 - \beta\sqrt{\dfrac{\gamma p_0}{\rho_0}}} \end{aligned} \tag{27}$$

Hence, for a 2-simple wave, the velocity and pressure are continuous, while the density and A are discontinuous. This is similar to a contact discontinuity in the Euler system. Here we call it a generalized contact discontinuity.

Since the 2-simple wave corresponds to $\mu_2 = (u - B)/A$, one would wonder whether μ_2 is continuous across a contact discontinuity. If it were not continuous, then the contact discontinuity

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would be like an expansion fan in the wave frame. Using Equation (27), we have

$$\frac{u-B}{A} \equiv \frac{(1-\alpha)u_0 - \beta\sqrt{\dfrac{\gamma p_0}{\rho_0}}}{A_0}$$

which means μ_2 is a constant, hence μ_2 is continuous across a contact discontinuity.

(3) For the third characteristic field, which is genuinely nonlinear as we have shown, we have

$$u = u_0 + \frac{2a_0}{\gamma-1}\left(\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - 1\right) = u_0 - \frac{2a_0}{\gamma-1} + \frac{2a}{\gamma-1} \tag{28}$$

$$a = a_0\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} \tag{29}$$

$$p = p_0\left(\frac{\rho}{\rho_0}\right)^{\gamma} \tag{30}$$

$$A = A_0\left(\frac{(1-\alpha)\left(u_0 - \dfrac{2a_0}{\gamma-1}\right) + a_0\left(\dfrac{2(1-\alpha)}{\gamma-1} + 1 - \beta\right)\left(\dfrac{\rho}{\rho_0}\right)^{(\gamma-1)/2}}{u_0(1-\alpha) + a_0(1-\beta)}\right)^{-\frac{2\alpha-\beta(1-\gamma)}{2(1-\alpha)+(\gamma-1)(1-\beta)}} \tag{31}$$

Consider a 3-simple wave centred at $(\lambda, \xi) = (0, 0)$. Since the second characteristic curve is a straight line in a 3-simple wave, we can solve

$$\frac{d\xi}{d\lambda} = \frac{u-B+a}{A}$$

to get

$$\frac{(1-\alpha)u + (1-\beta)a}{A} = \frac{\xi}{\lambda} \tag{32}$$

Inserting (28)–(31) into (32) yields the following equation for ρ/ρ_0

$$\frac{\rho}{\rho_0} = \left\{\frac{1}{\dfrac{2a_0(1-\alpha)}{\gamma-1} + a_0(1-\beta)}\left\{-(1-\alpha)\left(u_0 - \frac{2a_0}{\gamma-1}\right) + \left[A_0(u_0(1-\alpha) + a_0(1-\beta))^{[2\alpha-\beta(1-\gamma)]/[2(1-\alpha)+(\gamma-1)(1-\beta)]}\frac{\xi}{\lambda}\right]^{[2(1-\alpha)+(\gamma-1)(1-\beta)]/(1+\gamma)}\right\}\right\}^{2/(\gamma-1)} \tag{33}$$

Hence, with (33) and (28)–(31), we can obtain (ρ, u, p, A) at any point (ξ, λ) inside the third characteristic field.

Proposition 4
In the case of $A \equiv 1$, $B = 0$, (33) reduces to

$$\frac{\rho}{\rho_0} = \left\{ \frac{\gamma-1}{\gamma+1} \frac{1}{a_0} \left[u + a - \left(u_0 - \frac{2a_0}{\gamma-1} \right) \right] \right\}^{2/(\gamma-1)}$$

which reduces to the relation as in the Eulerian approach.

(4) The 4-simple wave is determined by

$$\frac{d\rho}{dA} = 0, \quad \frac{du}{dA} = 0, \quad \frac{dp}{dA} = 0$$

This means that in a 4-simple wave, the density, the velocity, and the pressure are all continuous, while A may have a discontinuity. We call such a discontinuity a motionless A-discontinuity since the corresponding eigenvalue is zero.

In summary, there are possibly 4 simple waves: left-going expansion fan (1-simple wave), generalized contact discontinuity, right-going expansion fan (3-simple wave), and motionless A-discontinuity.

3.2. Shock waves

The pure mathematical theory for shock waves can be found in References [9, 10]. Across a shock wave, not only the physical conservation law, but also the geometrical conservation law must satisfy the Rankine–Hugoniot relation. Now consider the jump relation in the transformed space (ξ, λ). If (17) is self-contained, then the Rankine–Hugoniot jump relation is

$$\langle F(W) \rangle = \sigma \langle W \rangle, \quad \xi = \xi_s \tag{34}$$

where ξ_s is the position of the discontinuity and σ is the speed of discontinuity in the transformed space.

Introducing $W = Jw$ and $F(W) = J(\xi_t w + \xi_x f)$ into (34), and noting that $J = A$ and $\xi_t = -B/A$, we obtain from (34)

$$\langle -Bw + f \rangle = \sigma \langle Aw \rangle \tag{35}$$

Let w_0 and w be the pre-shock and post-shock state as before. In Section 5 we will show that the jump relation in the wave frame is equivalent to the jump relation in the Eulerian system, provided that A satisfies the Rankine–Hugoniot relation

$$\langle -B \rangle = \sigma \langle A \rangle \tag{36}$$

and the shock speed σ is given by

$$\sigma = \frac{s' - B_0}{A_0}$$

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Hence, for the physical quantities, the jump relations are the same as for the Eulerian approach:

$$u = u_0 \pm \frac{a_0}{\gamma}(\varpi - 1)\left(\frac{2\gamma}{\gamma+1}\right)^{1/2}\left(\varpi + \frac{\gamma-1}{\gamma+1}\right)^{-1/2} \tag{37}$$

$$\frac{\rho}{\rho_0} = \frac{(\gamma+1)\varpi + \gamma - 1}{(\gamma-1)\varpi + \gamma + 1} \tag{38}$$

$$\frac{a}{a_0} = \sqrt{\varpi \cdot \frac{\rho_0}{\rho}} = \sqrt{\varpi \cdot \frac{(\gamma-1)\varpi + \gamma + 1}{(\gamma+1)\varpi + \gamma - 1}} \tag{39}$$

$$s' = u_0 \pm a_0\sqrt{\frac{(\varpi+1)\gamma + \varpi - 1}{2\gamma}} \tag{40}$$

where $\varpi = p/p_0$ is the shock intensity.

With (36) we have

$$\frac{A}{A_0} = 1 + \frac{\alpha(u_0 - u) + \beta(a_0 - a)}{u_0 \pm a_0\sqrt{\dfrac{(\varpi+1)\gamma + \varpi - 1}{2\gamma}} - (\alpha u_0 + \beta a_0)} \tag{41}$$

In (37), (40) and (41), the plus sign corresponds to a right-going shock wave while the minus sign corresponds to a left-going shock wave.

4. RIEMANN PROBLEM IN THE WAVE FRAME

A complete review for theories concerning Riemann problems can be found in the book by Serre [11], and a clear presentation suitable for numerical implementation in CFD is given by Toro [12]. The Riemann problem for general systems of conservation laws has been studied by Liu [13]. In the Eulerian system, it is well known that the solution of a Riemann problem for gas dynamics is composed of a left-going expansion fan (or shock wave), a contact discontinuity (in the middle), and a right-going shock wave (or expansion fan). In special cases there could be a vacuum state in the middle. In this paper, we do not consider such particular situations.

In the wave frame, we have another simple wave: the motionless A-discontinuity, which has a speed zero. By analogy with the Riemann problem in the Eulerian system, it is obvious that the solution is composed of a left-going expansion fan (or shock wave), a motionless A-discontinuity, a generalized contact discontinuity, and a right-going shock wave (or expansion fan). See Figure 1. Note that the generalized contact discontinuity may be to the left of the motionless A-discontinuity.

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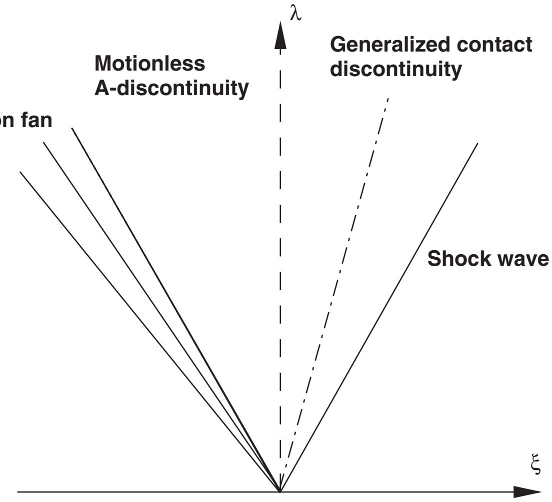

Figure 1. Riemann problem defined in the GCCS. The solution is composed of a left-going expansion fan (or shock wave), a motionless A-discontinuity, a generalized contact discontinuity, and a right-going shock wave (or expansion fan).

Let U_l and U_r be the two initial states of the Riemann problem. The four waves divide the flow regime into five uniform states, from left to right, they are, respectively, U_l, U_1, U_2, U_3, U_r, with

$$U_\mathrm{l}=\begin{pmatrix}\rho_\mathrm{l}\\ u_\mathrm{l}\\ p_\mathrm{l}\\ A_\mathrm{l}\end{pmatrix},\quad U_1=\begin{pmatrix}\rho_{m\mathrm{l}}\\ u_m\\ p_m\\ A_1\end{pmatrix},\quad U_2=\begin{pmatrix}\rho_{m\mathrm{l}}\\ u_m\\ p_m\\ A_2\end{pmatrix},\quad U_3=\begin{pmatrix}\rho_{m\mathrm{r}}\\ u_m\\ p_m\\ A_3\end{pmatrix},\quad U_\mathrm{r}=\begin{pmatrix}\rho_\mathrm{r}\\ u_\mathrm{r}\\ p_\mathrm{r}\\ A_\mathrm{r}\end{pmatrix}$$

if the generalized contact discontinuity is in the right of the motionless A-discontinuity ($u_m>0$), and

$$U_\mathrm{l}=\begin{pmatrix}\rho_\mathrm{l}\\ u_\mathrm{l}\\ p_\mathrm{l}\\ A_\mathrm{l}\end{pmatrix},\quad U_1=\begin{pmatrix}\rho_{m\mathrm{l}}\\ u_m\\ p_m\\ A_1\end{pmatrix},\quad U_2=\begin{pmatrix}\rho_{m\mathrm{r}}\\ u_m\\ p_m\\ A_2\end{pmatrix},\quad U_3=\begin{pmatrix}\rho_{m\mathrm{r}}\\ u_m\\ p_m\\ A_3\end{pmatrix},\quad U_\mathrm{r}=\begin{pmatrix}\rho_\mathrm{r}\\ u_\mathrm{r}\\ p_\mathrm{r}\\ A_\mathrm{r}\end{pmatrix}$$

if the generalized contact discontinuity is in the left of the motionless A-discontinuity ($u_m\leqslant 0$).

The problem seems to be much more complicated than the Riemann problem in the Eulerian system. However, as we will see, the procedure to determine the pressure p_m (and velocity u_m) is the same as in the Eulerian system, since it is independent of A, and B.

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If the 1-wave is a left-going expansion fan ($p_m \leqslant p_l$), then using (21) and (23), we have

$$u_m = u_l - \frac{2a_l}{\gamma - 1}\left[\left(\frac{p_m}{p_l}\right)^{(\gamma-1)/2\gamma} - 1\right] \tag{42}$$

If the 1-wave is a left-going shock wave ($p_m > p_l$), then using (37) we have

$$u_m = u_l - \frac{a_l}{\gamma}\left(\frac{p_m}{p_l} - 1\right)\left(\frac{\dfrac{2\gamma}{\gamma+1}}{\dfrac{p_m}{p_l} + \dfrac{\gamma-1}{\gamma+1}}\right)^{1/2} \tag{43}$$

For convenience, one can combine (42) and (43) to write

$$u_m = u_l - f_l(p_m, U_l) \tag{44}$$

where

$$f_l(p_m, U_l) = \begin{cases} \dfrac{2a_l}{\gamma - 1}\left[\left(\dfrac{p_m}{p_l}\right)^{(\gamma-1)/2\gamma} - 1\right], & p_m \leqslant p_l \\ \dfrac{a_l}{\gamma}\left(\dfrac{p_m}{p_l} - 1\right)\left(\dfrac{\dfrac{2\gamma}{\gamma+1}}{\dfrac{p_m}{p_l} + \dfrac{\gamma-1}{\gamma+1}}\right)^{1/2}, & p_m > p_l \end{cases} \tag{45}$$

If the 3-wave is a right-going expansion fan ($p_m \leqslant p_r$), then using (28) and (30), we have

$$u_m = u_r + \frac{2a_r}{\gamma - 1}\left[\left(\frac{p_m}{p_r}\right)^{(\gamma-1)/2\gamma} - 1\right] \tag{46}$$

If the 3-wave is a right-going shock wave ($p_m > p_r$), then using (37) we have

$$u_m = u_r + \frac{a_r}{\gamma}\left(\frac{p_m}{p_r} - 1\right)\left(\frac{\dfrac{2\gamma}{\gamma+1}}{\dfrac{p_m}{p_r} + \dfrac{\gamma-1}{\gamma+1}}\right)^{1/2} \tag{47}$$

For convenience, one can combine (46) and (47) to write

$$u_m = u_r + f_r(p_m, U_r) \tag{48}$$

where

$$f_r(p_m, U_r) = \begin{cases} \dfrac{2a_r}{\gamma - 1}\left[\left(\dfrac{p_m}{p_r}\right)^{(\gamma-1)/2\gamma} - 1\right], & p_m \leqslant p_r \\ \dfrac{a_r}{\gamma}\left(\dfrac{p_m}{p_r} - 1\right)\left(\dfrac{\dfrac{2\gamma}{\gamma+1}}{\dfrac{p_m}{p_r} + \dfrac{\gamma-1}{\gamma+1}}\right)^{1/2}, & p_m > p_r \end{cases} \tag{49}$$

Combining (44) and (48), we obtain the following nonlinear equation for p_m, which is exactly the same as in the Eulerian system, and the solution p_m is given by the root of the algebraic equation

$$u_{\mathrm{r}} - u_{\mathrm{l}} + f_{\mathrm{r}}(p_m, U_{\mathrm{r}}) + f_{\mathrm{l}}(p_m, U_{\mathrm{l}}) = 0 \tag{50}$$

The middle velocity follows as

$$u_m = \tfrac{1}{2}(u_{\mathrm{l}} + u_{\mathrm{r}}) + \tfrac{1}{2}(f_{\mathrm{r}}(p_m, U_{\mathrm{r}}) - f_{\mathrm{l}}(p_m, U_{\mathrm{l}})) \tag{51}$$

with p_m and u_m already known, now we can determine $\rho_{m\mathrm{l}}$, $\rho_{m\mathrm{r}}$, A_1, A_2, A_3.

Using (23), (30) and (38), we have

$$\frac{\rho_{m\mathrm{l}}}{\rho_{\mathrm{l}}} = \begin{cases} \left(\dfrac{p_m}{p_{\mathrm{l}}}\right)^{1/\gamma}, & p_m \leqslant p_{\mathrm{l}} \\ \dfrac{(\gamma+1)\dfrac{p_m}{p_{\mathrm{l}}} + \gamma - 1}{(\gamma-1)\dfrac{p_m}{p_{\mathrm{l}}} + \gamma + 1}, & p_m > p_{\mathrm{l}} \end{cases} \tag{52}$$

$$\frac{\rho_{m\mathrm{r}}}{\rho_{\mathrm{r}}} = \begin{cases} \left(\dfrac{p_m}{p_{\mathrm{r}}}\right)^{1/\gamma}, & p_m \leqslant p_{\mathrm{r}} \\ \dfrac{(\gamma+1)\dfrac{p_m}{p_{\mathrm{r}}} + \gamma - 1}{(\gamma-1)\dfrac{p_m}{p_{\mathrm{r}}} + \gamma + 1}, & p_m > p_{\mathrm{r}} \end{cases} \tag{53}$$

Using (24), (31) and (41), we have

$$\frac{A_1}{A_{\mathrm{l}}} = \begin{cases} \left(\dfrac{(1-\alpha)\left(u_{\mathrm{l}}+\dfrac{2a_{\mathrm{l}}}{\gamma-1}\right)-a_{\mathrm{l}}\left(\dfrac{2(1-\alpha)}{\gamma-1}+1+\beta\right)\left(\dfrac{\rho_{m\mathrm{l}}}{\rho_{\mathrm{l}}}\right)^{(\gamma-1)/2}}{u_{\mathrm{l}}(1-\alpha)+a_{\mathrm{l}}(1-\beta)}\right)^{-\frac{2\alpha+\beta(1-\gamma)}{2(1-\alpha)+(\gamma-1)(1+\beta)}}, & p_m \leqslant p_{\mathrm{l}} \\ 1+\dfrac{\alpha(u_{\mathrm{l}}-u_m)+\beta\left(a_{\mathrm{l}}-\sqrt{\dfrac{\gamma p_m}{\rho_{m\mathrm{l}}}}\right)}{u_{\mathrm{l}}-a_{\mathrm{l}}\sqrt{\dfrac{\left(\dfrac{p_m}{p_{\mathrm{l}}}+1\right)\gamma+\dfrac{p_m}{p_{\mathrm{l}}}-1}{2\gamma}}-(\alpha u_{\mathrm{l}}+\beta a_{\mathrm{l}})}, & p_m > p_{\mathrm{l}} \end{cases} \tag{54}$$

$$\frac{A_3}{A_{\mathrm{r}}} = \begin{cases} \left(\dfrac{(1-\alpha)\left(u_{\mathrm{r}}-\dfrac{2a_{\mathrm{r}}}{\gamma-1}\right)+a_{\mathrm{r}}\left(\dfrac{2(1-\alpha)}{\gamma-1}+1-\beta\right)\left(\dfrac{\rho_{m\mathrm{r}}}{\rho_{\mathrm{r}}}\right)^{(\gamma-1)/2}}{u_{\mathrm{r}}(1-\alpha)+a_{\mathrm{r}}(1-\beta)}\right)^{-\frac{2\alpha-\beta(1-\gamma)}{2(1-\alpha)+(\gamma-1)(1-\beta)}}, & p_m \leqslant p_{\mathrm{r}} \\ 1+\dfrac{\alpha(u_{\mathrm{r}}-u_m)+\beta\left(a_{\mathrm{r}}-\sqrt{\dfrac{\gamma p_m}{\rho_{m\mathrm{r}}}}\right)}{u_{\mathrm{r}}+a_{\mathrm{r}}\sqrt{\dfrac{\left(\dfrac{p_m}{p_{\mathrm{r}}}+1\right)\gamma+\dfrac{p_m}{p_{\mathrm{r}}}-1}{2\gamma}}-(\alpha u_{\mathrm{r}}+\beta a_{\mathrm{r}})}, & p_m > p_{\mathrm{r}} \end{cases} \tag{55}$$

The state A_2 depends on whether the generalized contact discontinuity is on the right ($u_m>0$) or left ($u_m \leqslant 0$) of the motionless A-discontinuity. Using (27), we have

$$A_2 = A_3 \frac{(1-\alpha)u_m - \beta\sqrt{\dfrac{\gamma p_m}{\rho_{m\text{l}}}}}{(1-\alpha)u_{m\text{l}} - \beta\sqrt{\dfrac{\gamma p_m}{\rho_{m\text{r}}}}} \quad \text{if } u_m>0 \tag{56}$$

$$A_2 = A_1 \frac{(1-\alpha)u_m - \beta\sqrt{\dfrac{\gamma p_m}{\rho_{m\text{r}}}}}{(1-\alpha)u_m - \beta\sqrt{\dfrac{\gamma p_m}{\rho_{m\text{l}}}}} \quad \text{if } u_m\leqslant 0 \tag{57}$$

Therefore, the five uniform states of the flow regime divided by four waves can be obtained as above. In Section 3.1, we have already shown that we could get (ρ, u, p, A) at any point (ξ, λ) inside the left and right simple waves. Thus, the Riemann problem in the wave frame can be solved exactly.

5. EQUIVALENCE OF EXPANSION FANS IN THE EULERIAN SYSTEM AND IN THE WAVE FRAME

The equivalence of expansion waves in moving frames (not including the wave frame) has been studied in Reference [6]. Here we extend the results to the wave frame.

A physical expansion fan is an expansion fan in the Eulerian coordinate system. A left-going expansion fan is defined by (4)–(5) and a right-going expansion fan is defined by (8)–(9). We only consider a left-going expansion fan in the subsequent analysis. We are interested here in whether a physical expansion fan remains to be an expansion wave or is changed into another wave. For simplicity, we only consider a left-going expansion fan centred at $(0, 0)$ so that

$$u(x,t) = \frac{2}{\gamma+1}\left[R^+ + \frac{x}{t}\right] \tag{58}$$

$$a(x,t) = \frac{2}{\gamma+1}\left[R^+ - \frac{\gamma-1}{2}\frac{x}{t}\right] \tag{59}$$

The wave speed for a left-going physical expansion fan is given by

$$\mu = u - a$$

and the corresponding eigenvalue in the wave frame is

$$\mu = \frac{u-B-a}{A} = \frac{(1-\alpha)u - (1+\beta)a}{A}$$

In order to find the sign of

$$\frac{\partial \mu}{\partial \xi} = \frac{(1-\alpha)\dfrac{\partial u}{\partial \xi} - (1+\beta)\dfrac{\partial a}{\partial \xi}}{A} - \frac{(1-\alpha)u-(1+\beta)a}{A^2}\frac{\partial A}{\partial \xi} \tag{60}$$

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or

$$\frac{\partial \mu}{\partial \xi} = \frac{\partial \mu}{\partial x} x_\xi + \frac{\partial \mu}{\partial t} t_\xi = -\frac{(1-\alpha)u - (1+\beta)a}{A}\frac{\partial A}{\partial x} + (1-\alpha)\frac{\partial u}{\partial x} - (1+\beta)\frac{\partial a}{\partial x} \tag{61}$$

we must know the exact expressions for u, a, B and A. The formulas for u, a, and hence $B = \alpha u + \beta a$, are already given by (4) and (5). It remains to find A, which can be obtained by solving the geometrical conservation law (15).

Now we want to determine A in two ways: one in the physical space and one in the wave frame. First consider the wave frame so that A is determined by (15). Using (58) and (59) we obtain

$$\frac{\partial B}{\partial \xi} = \alpha \frac{\partial u}{\partial \xi} + \beta \frac{\partial a}{\partial \xi} = \frac{\varepsilon A}{t}$$

where

$$\varepsilon = \frac{2\alpha}{\gamma+1} - \frac{\gamma-1}{\gamma+1}\beta \tag{62}$$

Hence, (15) reduces to $A_\lambda - \varepsilon A/t = 0$, or $t = \varepsilon A/A_\lambda$, which, on using (12), gives

$$\mathrm{d}\lambda = \varepsilon\, \mathrm{d}\left(\frac{A}{A_\lambda}\right) = \varepsilon \frac{\partial}{\partial \lambda}\left(\frac{A}{A_\lambda}\right)\mathrm{d}\lambda + \varepsilon \frac{\partial}{\partial \xi}\left(\frac{A}{A_\lambda}\right)\mathrm{d}\xi$$

so that

$$\varepsilon \frac{\partial}{\partial \lambda}\left(\frac{A}{A_\lambda}\right) = 1 \tag{63}$$

$$\varepsilon \frac{\partial}{\partial \xi}\left(\frac{A}{A_\lambda}\right) = 0 \tag{64}$$

Equation (64) means that A has the following form $A = \Phi(\lambda)\Psi(\xi)$. Inserting this form into (63) yields

$$\varepsilon \frac{\partial}{\partial \lambda}\left(\frac{\Phi}{\Phi_\lambda}\right) = 1$$

so that

$$\Phi = c_1(\varepsilon^{-1}\lambda + c)^{\varepsilon}$$

where c and c_1 are two arbitrary constants. Hence, the general solution of (15) is given by

$$A(\xi, \lambda) = \Psi(\xi)\left(\frac{\lambda}{\dfrac{2\alpha}{\gamma+1} - \dfrac{\gamma-1}{\gamma+1}\beta} + c\right)^{([2\alpha/(\gamma+1)]-[(\gamma-1)/(\gamma+1)]\beta)} \tag{65}$$

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where we have combined the constant c_1 with $\Psi(\xi)$, which depends on the initial data for $A(\xi,\lambda)$. One perhaps chooses a constant initial value $A(\xi,\lambda)=A(\xi,0)$, so that $\Psi(\xi)\, c^{([2\alpha/(\gamma+1)]-[(\gamma-1)/(\gamma+1)]\beta)}=A_0$, or

$$A(\xi,\lambda)=A_0\left(\frac{c^{-1}\lambda}{\dfrac{2\alpha}{\gamma+1}-\dfrac{\gamma-1}{\gamma+1}\beta}+1\right)^{([2\alpha/(\gamma+1)]-[(\gamma-1)/(\gamma+1)]\beta)} \tag{66}$$

for an arbitrary c.

If $A(\xi,\lambda)$ is constant at $\lambda=0$, then, by (60), we have

$$\frac{\partial\mu}{\partial\xi}=\frac{1}{t}\left(1+\frac{\beta(\gamma-1)-2\alpha}{\gamma+1}\right) \tag{67}$$

since

$$\frac{\partial}{\partial\xi}\left(\frac{x}{t}\right)=\frac{1}{t}\frac{\partial x}{\partial\xi}-\frac{x}{t^2}\frac{\partial t}{\partial\xi}=\frac{A}{t}$$

A close examination of (67) leads to

Proposition 5

Let the initial data for $A(\xi,\lambda)$ be constant, i.e. $A(\xi,0)=\text{Const}$, then, in the wave frame, the physical expansion fan remains to be an expansion wave if $\alpha<(\gamma+1)/2+[\beta(\gamma-1)]/2$, degenerates to a linear wave if $\alpha=(\gamma+1)/2+[\beta(\gamma-1)]/2$, and changes to a compression wave if $\alpha>(\gamma+1)/2+[\beta(\gamma-1)]/2$.

Now let us determine A in the physical space. Noting that

$$A_\lambda=A_t+A_x x_\lambda=A_t+BA_x$$

$$B_\xi=B_x x_\xi=AB_x$$

we can rewrite (15) as

$$\frac{\partial A}{\partial t}+B\frac{\partial A}{\partial x}=A\frac{\partial B}{\partial x} \tag{68}$$

which, when replacing B by $B=\alpha u+\beta a$, yields

$$\frac{\partial A}{\partial t}+(\alpha u+\beta a)\frac{\partial A}{\partial x}=\left(\alpha\frac{\partial u}{\partial x}+\beta\frac{\partial a}{\partial x}\right)A$$

or, when u and a are replaced by (58) and (59), respectively,

$$\frac{\partial\ln A}{\partial t}+\frac{2}{\gamma+1}\left((\alpha+\beta)R^+ +\frac{x}{t}\left(\alpha-\frac{\gamma-1}{2}\beta\right)\right)\frac{\partial\ln A}{\partial x}$$
$$=\frac{2}{\gamma+1}\left(\alpha-\frac{\gamma-1}{2}\beta\right)\frac{1}{t} \tag{69}$$

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The solution of (69) can be expressed as

$$\ln A = Q + \frac{2}{\gamma+1}\left(\alpha - \frac{\gamma-1}{2}\beta\right)\ln t \tag{70}$$

where Q satisfies the homogeneous equation

$$\frac{\partial Q}{\partial t} + \left(\phi + \varphi\frac{x}{t}\right)\frac{\partial Q}{\partial x} = 0$$

where

$$\phi = \frac{2}{\gamma+1}(\alpha+\beta)R^{+} \tag{71}$$

$$\varphi = \frac{2}{\gamma+1}\left(\alpha - \frac{\gamma-1}{2}\beta\right) \tag{72}$$

Using the method of characteristics, the solution for $Q(t,x)$ is given by

$$Q(t,x) = F(\eta) \tag{73}$$

with

$$\eta = \frac{x}{t^{\varphi}} + \frac{\phi}{(\varphi-1)}\frac{t}{t^{\varphi}}$$

The exact form for $F(\eta)$ is to be determined by the initial condition for A.

Substituting (73) into (70) yields

$$A(x,t) = t^{\varphi}\exp F(\eta) \tag{74}$$

Obviously, the solution for $A(x,t)$ is not an auto-similar function (of x/t) inside the expansion wave, while the solutions for u and a are. If we choose $F(\eta) = \ln\eta$, then

$$A(x,t) = x + \frac{\phi}{(\varphi-1)}t \tag{75}$$

which correspond to $A(x,0) = x$. Otherwise, it is not so easy to relate $A(x,0)$ to F.

With $A(x,t)$ given by (75), and with u and a given by (4) and (5), respectively, we obtain, from (61),

$$\frac{\partial\mu}{\partial\xi} = 0$$

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Proposition 6
If $A(x,0)=x$ so that $A(x,t)$ is given by (75), then a left-going expansion fan in the physical space (x,t) degenerates into a linearly degenerate wave in the transformed space (ξ, λ).

6. EQUIVALENCE OF SHOCK WAVES IN THE EULERIAN SYSTEM AND IN THE WAVE FRAME

The following theorem, which now holds for the wave frame, has been proved in Reference [7] for the unified coordinate system of Hui,

Proposition 7
If A satisfies the Rankine–Hugoniot relation

$$\langle -B\rangle = \sigma\langle A\rangle \tag{76}$$

then the jump relation in the physical space (11) and the jump relation in the transformed space (35) are equivalent, with σ given by

$$\sigma = \frac{s' - B_{\mathrm{l}}}{A_{\mathrm{l}}} \tag{77}$$

From (76), we have, for continuous h,

$$A_{\mathrm{r}} = A_{\mathrm{l}}\left(1 + \frac{B_{\mathrm{l}} - B_{\mathrm{r}}}{s' - B_{\mathrm{l}}}\right) = A_{\mathrm{l}} S(s, M, \alpha, \beta) \tag{78}$$

where

$$S(s, M, \alpha, \beta) = 1 + \frac{\alpha\left(\frac{M-s}{M} - \frac{(\gamma-1)(M-s)^2+2}{(\gamma+1)(M-s)M}\right)M + \beta\left(1 - \sqrt{\frac{2\gamma(M-s)^2-(\gamma-1)}{\gamma+1}\,\frac{(\gamma-1)(M-s)^2+2}{(\gamma+1)(M-s)^2}}\right)}{s - \alpha M - \beta}$$

The transformation between the wave frame and the original Eulerian coordinate system is invertible if and only if

$$S(s, M, \alpha, \beta) > 0 \tag{79}$$

In Reference [7] we have proved the following.

Proposition 8
Consider the unified coordinate system of Hui *et al.* with $\alpha = h$ and $\beta = 0$. If the speed of a physically relevant shock wave, with $s>0$, satisfies the constraint

$$0 < s < M - 1 \tag{80}$$

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then the function $S(h)$ becomes negative for h satisfying

$$0<h_b<h<h_a<1$$

where

$$h_a = \frac{1}{\frac{(\gamma-1)\frac{M}{s}+2}{(\gamma+1)} + \frac{2}{(\gamma+1)(M-s)s}}, \quad h_b = \frac{s}{M}$$

In other words, there is a parameter range $h \in (0, 1)$ such that the unified coordinate system of Hui *et al.* is not invertible.

Now we want to consider the wave frame. We want to find the set of parameters (α, β) such that $S(s, M, \alpha, \beta)<0$. A direct calculation shows that $S=0$ at $\alpha=\alpha_a$ with

$$\alpha_a = \frac{(\gamma+1)s(M-s) - \beta(\gamma+1)(M-s)\sqrt{\left(\frac{2\gamma(M-s)^2-(\gamma-1)}{\gamma+1}\frac{(\gamma-1)(M-s)^2+2}{(\gamma+1)(M-s)^2}\right)}}{((\gamma-1)M+2s)(M-s)+2} \tag{81}$$

There is also a singular point for S, i.e. $S^{-1}=0$ at $\alpha=\alpha_b$, with

$$\alpha_b = \frac{s-\beta}{M} \tag{82}$$

Thus we have the following proposition.

Proposition 9

Consider the wave frame. The function $S=S(s, M, \alpha, \beta)$ becomes negative for α satisfying

$$\alpha_b<\alpha<\alpha_a$$

In other words, there is a parameter range for α such that the wave frame is not invertible.

7. SUMMARY AND PRELIMINARY NUMERICAL RESULTS

7.1. Summary

In this paper, we have studied compressible flow equations in a wave frame which moves at the characteristic wave speed $B=\alpha u+\beta a$. The question of hyperbolicity, simple waves, weak solution, Riemann problems, and equivalence of waves in different coordinate systems are analysed. Among various results we may emphasize the following remarks:

(1) there is a restriction on the choice of α and β. If we perturb the Lagrangian system by replacing its speed with $B=u+\varepsilon a$ where ε is a small parameter, then strong hyperbolicity can be recovered.
(2) there is a new simple wave which is motionless and for which all the flow parameters are continuous across the wave but A is discontinuous.

(3) wave types depend on the choice of coordinate systems. For instance, an expansion wave seen in one frame may become a compression wave in another frame.

Apart from its theoretical interest, we believe that the wave frame can be used for problems where expansion fans need be captured with a special accuracy in computational fluid dynamics. With the current CFD algorithms all the methods work well for computing shock waves with moderate speeds, while the Lagrangian approach has advantages over the Eulerian approach for computing contact discontinuities. Without special numerical treatment, it seems that no method works very well for expansion waves. The reason could be that expansion waves move at the largest characteristic speed (convective speed u plus the sound speed a) or at the smallest characteristic speed (convective speed u minus the sound speed a). By intuition from the advantage of the Lagrangian coordinate system for computing contact discontinuities, it is expected that expansion fans can be more exactly computed by using a frame that follows the expansion waves.

7.2. One-dimensional result

Here we provide a preliminary result for one-dimensional study. We have built a Godunov scheme for the compressible flow equations in the wave frame. In the computation we have used $\gamma = 1.4$ for the ratio of specific heats of the gas. The initial data consist of two constant states (ρ_l, u_l, p_l, A_l) and (ρ_r, u_r, p_r, A_r), separated by a discontinuity at $x = 0.3$, with $A_l = A_r = 1$. The spatial domain

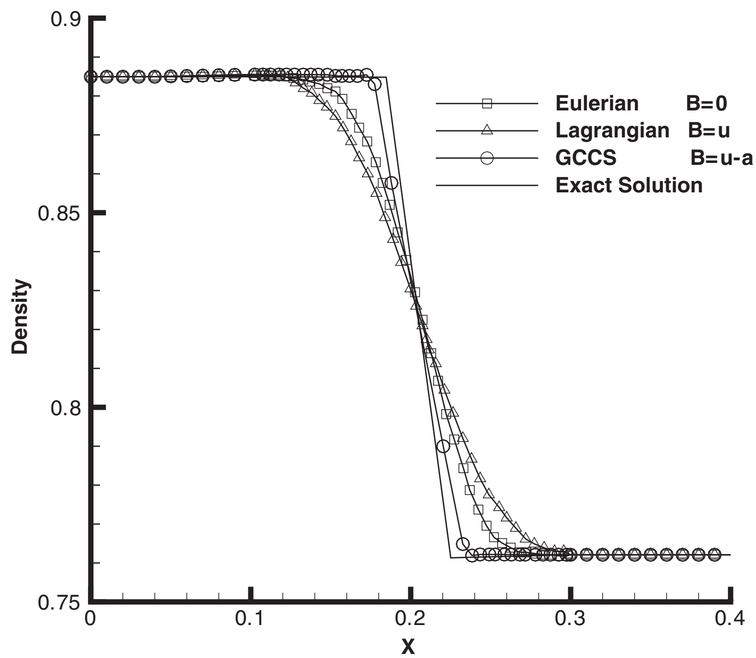

Figure 2. Computed density using Godunov scheme in three different coordinates compared with exact solution at output time $t = 0.2$. It should emphasized that we have used the same discretization for the three different frames. The advantage of the wave frame in capturing expansion fans is obvious.

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is the interval [0, 1] which is discretized with 100 computing cells. The CFL number used to determine the size of the time step is 0.5. Only the results for density are shown. To demonstrate the advantage of the wave frame (GCCS) the initial states (ρ, u, p) are (0.885, 0.577, 0.843) on the left and (0.762, 0.748, 0.684) on the right which result in a fast left-going expansion fan. The frame is defined by $B = u - a$. Hence, the frame moves exactly at the speed of the expansion fan. It could be seen by the result in Figure 2 that the expansion fan is more exactly computed by the wave frame (denoted as GCCS). The Lagrangian method, which resolves the contact discontinuity sharply though, gives the worst result. It must be emphasized that for the comparison of different coordinate systems we have here used the same grid and only first-order accurate spatial discretization.

7.3. Two-dimensional result

The GCCS presented in this paper can be extended to two dimensions straightforwardly in the same way as the unified coordinate system [3]. Here it is not our objective to present the numerical

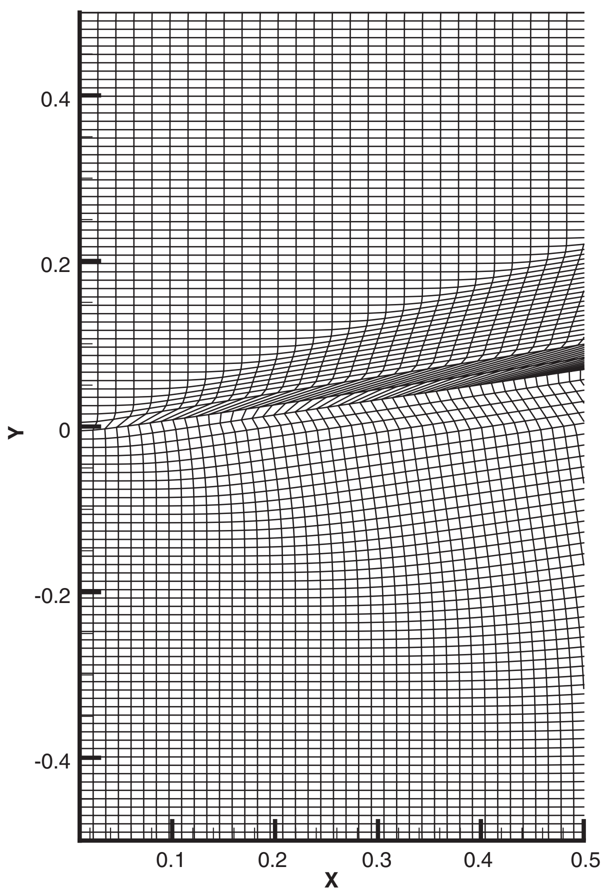

Figure 3. Mesh for the two-dimensional Riemann problem.

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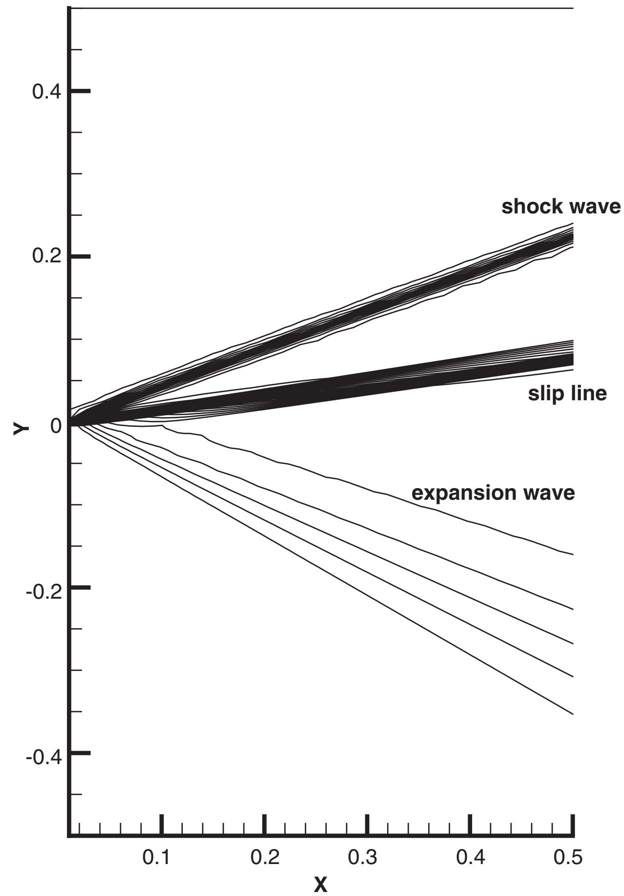

Figure 4. Mach contours for the two-dimensional Riemann problem.

details of the GCCS approach, rather we simply show the numerical results for a two-dimensional Riemann problem.

The computation domain is defined by $(0, 0.5) \times (-0.5, 0.5)$, a grid number of 50×100 is used in the case of Eulerian computation. The left boundary condition is $(\rho = 0.42, p = 0.21, M = 4)$ for $y > 0$ and $(\rho = 1, p = 1, M = 1.5)$ for $y < 0$. In the GCCS method, the mesh is allowed to move at the speed $\mathbf{V}(1 - 0.8/M)$, where M is the local Mach number. The corresponding mesh and the Mach contours obtained by the GCCS method are displayed in Figures 3 and 4, respectively.

In Figures 5–8, we displayed the density distributions along a vertical line located at $x = 0.4$, obtained by various coordinate systems. It must be emphasized here that we have only used the first-order Godunov method for all the coordinate systems and a simple use of different coordinate system leads to quite different resolution.

It is clear that the Eulerian approach yields poor results for the slipline and the expansion fan. One may use very high-order schemes to obtain better results, but this is not the objective of the present paper.

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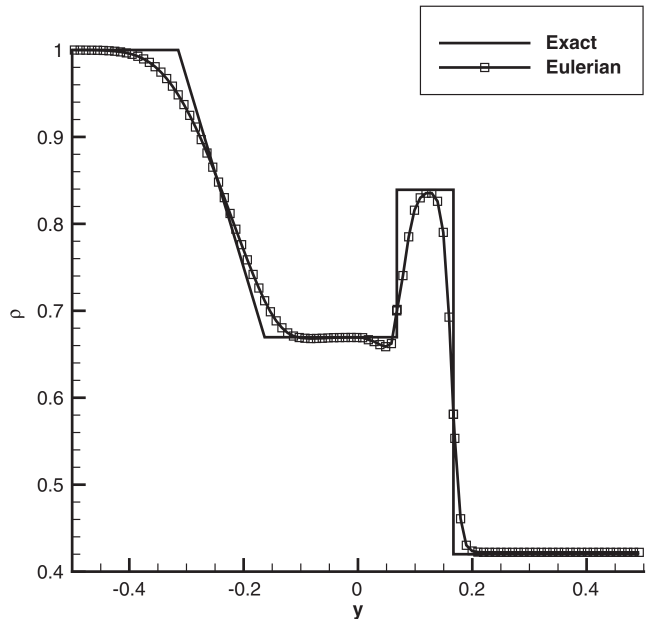

Figure 5. Density distribution along a vertical line, by the Eulerian method.

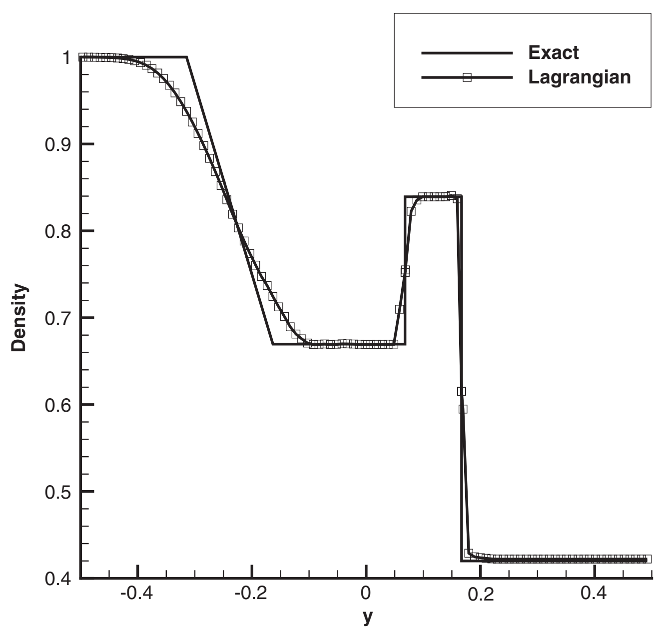

Figure 6. Density distribution along a vertical line, by the Lagrangian method.

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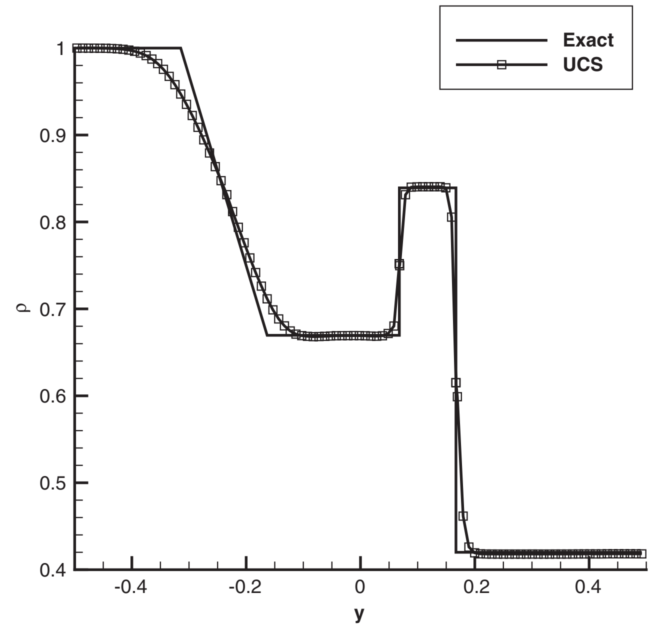

Figure 7. Density distribution along a vertical line, by the unified coordinate system method.

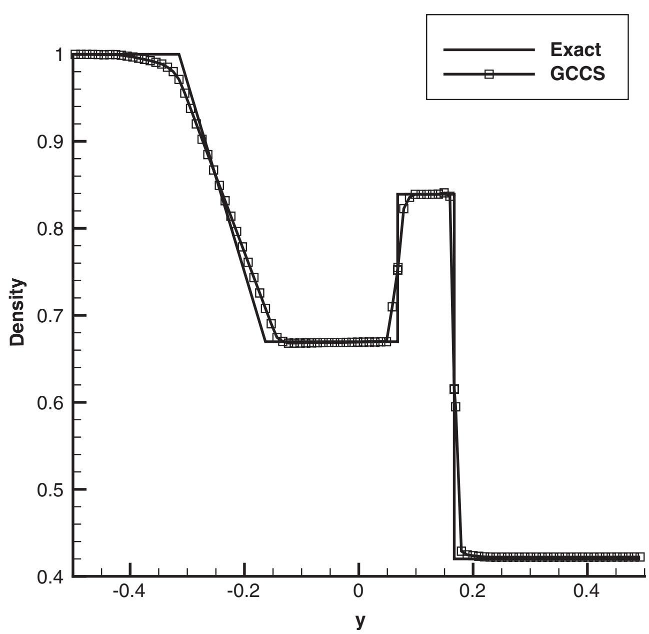

Figure 8. Density distribution along a vertical line, by the present generalized characteristic coordinate system method.

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The Lagrangian approach and the UCS approach both improve the slipline resolution, but yield a much poor result for the expansion fan.

The GCCS approach, however, captures well all the waves, including the shock wave, slipline, and expansion fan. Notably, it gives much better results for the expansion fan, while maintaining good results for the shock wave and slipline.

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